IEB Differential calculus practice — Memorandum

Every answer below was confirmed by an independent verifier at the moment the question was made.

The derivative from first principles

1

(a) Answer: $f'(x) = -4x + 4$

1. $f(x+h) = -2(x+h)^2 + 4(x+h) + 6$
2. $= -2x^2 - 4xh - 2h^2 + 4x + 4h + 6$
3. $f(x+h) - f(x) = -4xh - 2h^2 + 4h$
4. $f'(x) = \lim_{h \to 0} \frac{-4xh - 2h^2 + 4h}{h}$
5. $= \lim_{h \to 0} (-4x - 2h + 4)$
6. $= -4x + 4$

(b) Answer: $\frac{dy}{dx} = (\frac{3}{2})x^{1/2} - 3x^{-1/2} + 2x^{-3/2}$

1. $y = x^{3/2} - 6x^{1/2} - 4x^{-1/2}$
2. $\frac{dy}{dx} = (\frac{3}{2})x^{1/2} - 3x^{-1/2} + 2x^{-3/2}$

(c) Answer: $y = 12x + 14$

1. $f(-2) = -10$, so the point is $(-2; -10)$
2. $m = f'(-2) = 12$
3. $y + 10 = 12(x + 2)$
4. $y = 12x + 14$

(d) Answer: -2

1. $f(1) = 8$ and $f(2) = 6$
2. $\frac{6 - (8)}{2 - (1)} = -2$

2

(a) Answer: $4x + 2h - 2$

1. $f(x+h) = 2(x+h)^2 - 2(x+h) + 1$
2. $f(x+h) - f(x) = 4xh + 2h^2 - 2h$
3. $\frac{4xh + 2h^2 - 2h}{h} = 4x + 2h - 2$

(b) Answer: $f'(x) = 4x - 2$

1. $f'(x) = \lim_{h \to 0} (4x + 2h - 2)$
2. $= 4x - 2$

(c) Answer: $(-2; 17)$; $k = 3$

1. $g'(x) = 2x - 3 = -7$
2. $x = -2$
3. $g(-2) = 17$, so the point is $(-2; 17)$
4. $17 = -7(-2) + k$, so $k = 3$

(d) Answer: $(-1; 5)$

1. $f'(x) = 4x - 2 = -6$
2. $x = -1$
3. $f(-1) = 5$

Stationary points

3

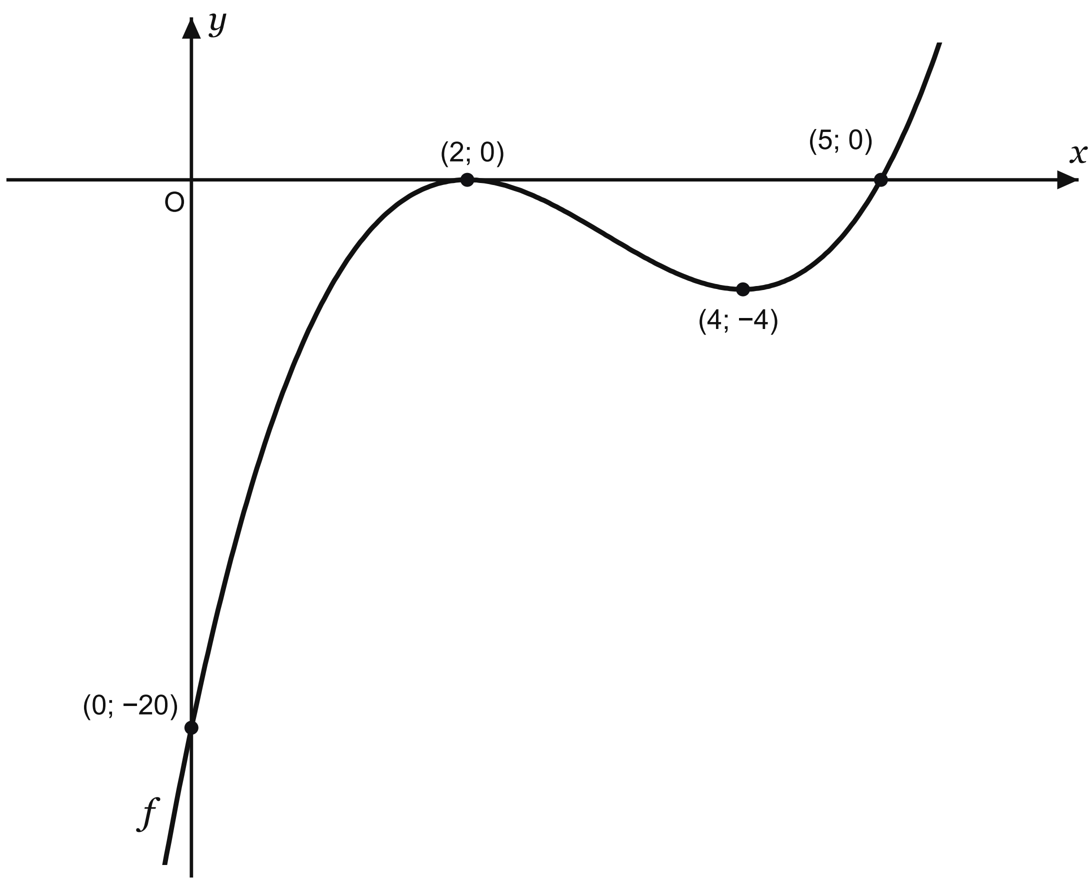
(a) Answer: (2; 0) and (5; 0)

1. $f(5) = 0$, so $(x - 5)$ is a factor
2. $f(x) = (x - 5)(x - 2)^2$
3. (2; 0) and (5; 0)

(b) Answer: (2; 0) and $(4; -4)$

1. $f'(x) = 3x^2 - 18x + 24$
2. $3x^2 - 18x + 24 = 0$
3. $x = 2$ or $x = 4$
4. (2; 0) and $(4; -4)$

(c) Answer: See the sketch below.



1. y-intercept $(0; -20)$; x-intercepts (2; 0), (5; 0)

2. turning points $(2; 0)$, $(4; -4)$

(d) Answer: $x > 3$

1. $f''(x) = 6x - 18$
2. $f''(x) > 0$: $6x - 18 > 0$
3. $x > 3$

(e) Answer: $x < 2$ or $x > 4$

1. $f'(x) > 0$
2. $x < 2$ or $x > 4$

(f) Answer: $-4 < k < 0$

1. the line $y = k$ must cut the graph three times: between the turning values
2. $-4 < k < 0$

4

(a) Answer: $d = -16$

1. $f(4) = 0$
2. $16 + d = 0$
3. $d = -16$

(b) Answer: (2; 4)

1. $f'(x) = 3x^2 - 18x + 24 = 0$
2. $3(x - (2))(x - (4)) = 0$
3. $x = 2$
4. $f(2) = 4$

(c) Answer: $x = 3$

1. $f''(x) = 6x - 18 = 0$
2. $x = 3$

(d) Answer: $k < 0$ or $k > 4$

1. the turning values are 0 and 4
2. one root when the line $y = k$ passes above the maximum or below the minimum
3. $k < 0$ or $k > 4$

(e) Answer: $y = 24x - 16$

1. $f(0) = -16$ and $f'(0) = 24$
2. $y = 24x - 16$

Optimisation

5

(a) Answer: See the working below.

1. the base is $(36 - 2x)$ by $(36 - 2x)$ and the height is x
2. $\mathrm{V} = x(36 - 2x)^2 = 4x^3 - 144x^2 + 1296x$

(b) Answer: $x = 6$ cm; $\mathrm{V} = 3456$ cm³

1. dV$/dx = 12x^2 - 288x + 1296 = 0$
2. $(6x - 36)(2x - 36) = 0$
3. $x = 6$ ($x = 18$ gives no box)
4. $\mathrm{V} = 6(36 - 12)^2 = 3456$ cm³

(c) Answer: -132 cm³$/cm$

1. dV$/dx$ at $x = 17$
2. $= -132$ cm³$/cm$

6

(a) Answer: See the working below.

1. $h = 12 - r$
2. $\mathrm{V} = (\frac{1}{3})\pi r^2(12 - r) = (\pi/3)(12r^2 - r^3)$

(b) Answer: $r = 8$ cm; $\mathrm{V} = 256\pi/3$ cm³

1. d$\mathrm{V}/dr = (\pi/3)(24r - 3r^2) = 0$
2. $r(24 - 3r) = 0$
3. $r = 8$ ($r = 0$ gives no cone)
4. $\mathrm{V} = (\pi/3)(12(8)^2 - (8)^3) = 256\pi/3$ cm³

(c) Answer: $12\pi\ cm^3/cm$

1. dV/dr at $r = 6$
2. $= 12\pi\ cm^3/cm$

Reading the graph of f'

7

(a) Answer: $-5 < x < 1$

1. f is increasing where $f'(x) > 0$: where the graph of f' lies above the x-axis
2. $-5 < x < 1$

(b) Answer: $x = 1$ (local maximum) and $x = -5$ (local minimum)

1. $f'(x) = 0$ at $x = -5$ and $x = 1$
2. f' changes from positive to negative at $x = 1$: a maximum

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(c) Answer: $x = -2$

1. $f''(x) = 0$ where f' has its turning point
2. $x = \frac{-5+1}{2} = -2$

(d) Answer: $x < -2$

1. f is concave up where $f''(x) > 0$: where f' is increasing
2. $x < -2$

(e) Answer: 5

1. the gradient of f at $x = 0$ is $f'(0) = 5$

8

(a) Answer: $f'(x) = 3x^2 + 6x + 3$

1. $f'(x) = a(x - (-1))^2$, touching the x-axis at C
2. $3 = a(-1)^2$, so $a = 3$
3. $f'(x) = 3x^2 + 6x + 3$

(b) Answer: $f(x) = x^3 + 3x^2 + 3x - 1$

1. $f(x) = ax^3 + bx^2 + cx + d$ with $f'(x) = 3ax^2 + 2bx + c = 3x^2 + 6x + 3$
2. $a = 1$, $b = 3$, $c = 3$
3. $f(0) = d = -1$
4. $f(x) = x^3 + 3x^2 + 3x - 1$

(c) Answer: A point of inflection

1. $f'(-1) = 0$, but f' does not change sign at $x = -1$
2. so f has a horizontal point of inflection there

(d) Answer: $x = -3$ or $x = 1$

1. $3(x - (-1))^2 = 12$
2. $(x - (-1))^2 = 4$
3. $x = -3$ or $x = 1$